Chapter 1

Introduction

1.1 Overview of Wavelet Research

Wavelet theory is a synthesis of ideas from many different research domains. It combines aspects like filter banks, multi-resolution, (bi)orthogonality, ...

The first wavelet transforms were continuous wavelet transforms. In the Continuous Wavelet Transform (CWT), a signal is analyzed by calculating inner products with test functions. All test functions are generated from one oscillating function with finite support, called a wavelet function. Only one condition is imposed on the wavelet function: its integral must be zero. Different test functions are created by translating and dilating the wavelet function, thus one has a customizable time/frequency window through which the analysis is done. This locality in both time and frequency is advantageous, compared to Fourier analysis [21].

The Discrete Wavelet Transform (DWT) is an adaptation to digital signals, which are discrete [43]. Usually not only the translations are limited to multiples of the sampling distance, but also the dilation (scale) factor becomes dyadic: one can distinguish different resolution levels, with a scale factor of two in between. Hence we have a multi-resolution analysis [34], which is related to pyramidal schemes like the Laplacian pyramid [1].

If the wavelet functions are orthogonal, the DWT is stable. The first orthogonal wavelets with compact support were constructed by Daubechies [12]. These allowed for an efficient implementation, the Fast Wavelet Transform (FWT) or the filter bank algorithm, creating the link with filter banks and Quadrature Mirror Filters (QMF), which are commonly used in subband coding. By relaxing the orthogonality conditions, biorthogonal wavelets could provide for both compact support and symmetry, as is shown by e.g. the biorthogonal wavelets constructed by Cohen, Daubechies and Feauveau [9].

Wavelet packets are an extension of wavelet theory: they provide an adapted basis using the filter bank algorithm, filling the gap between the standard wavelet decomposition tree and the area of subband coding.

The lifting scheme, developed by Sweldens, started as a method to improve a given discrete wavelet transform to obtain specific properties [48]. Later it became an efficient algorithm to calculate any wavelet transform as a sequence of simple lifting steps [14]. Digital signals are usually a sequence of integer numbers, while wavelet transforms result in floating point numbers. For an efficient reversible implementation it is of great importance to have a transform algorithm that converts integers to integers. Fortunately a lifting step can be modified to operate on integers, while preserving the reversibility [6]. Thus the lifting scheme became a method to implement reversible integer wavelet transforms.

The lifting scheme was also extended to a generic method to construct second-generation wavelets [49]. These are wavelets that are no longer created as the translation and dilation of one wavelet function. The lifting scheme allows to construct wavelets for higher dimensional grids, irregular samplings, triangularizations, spheres, … [40, 31].

Wavelets are used in many applications, like signal and image processing, image and video compression, mathematical physics, numerical analysis, communication, system theory, identification, geometrical modeling, computer graphics, theoretical physics, approximation theory, stochastic processes, …. Of course this list is far from exhaustive. We refer to the literature for more information, e.g. [3, 7, 8, 28, 29, 33, 36, 46, 58].

We focused on two applications: image processing and compression to create low-dimensional models for dynamical systems.

We worked out the details of integer wavelet transforms, based on the lifting scheme, for a class of biorthogonal wavelets (Cohen-Daubechies-Feauveau). Based on this we designed and implemented a software library called *WAILI* (Wavelets with Integer Lifting) that provides wavelet transforms and wavelet-based image processing operations on two-dimensional images [56]. Later we added support for very large images and block-based processing [55].

We created a new kind of second-generation wavelets on a rectangular grid, based on a red-black blocking scheme. These Red-Black wavelets are less anisotropic than tensor product wavelets [52].

We reduced the complexity of the Proper Orthogonal Decomposition (POD) by biorthogonal wavelet packet compression and evaluated the resulting Approximate POD by analyzing a large-scale dynamical system, described by partial differential equations [53].

1.2 Overview of the Thesis

In Chapter 2 we give an introduction to the concepts multi-resolution analysis and wavelets from the classical viewpoint. The lifting scheme, discussed in Chapter 3, not only provides a framework to create so-called second-generation wavelets, but also allows to implement classical wavelet transforms. It has several advantages over the classical computation schemes and provides additional features like a true integer transform. We have worked out the details of integer wavelet transforms for a class of biorthogonal wavelets (Cohen-Daubechies-Feauveau)

The largest share of this thesis is about the application of wavelets in image processing. Many image processing operations can be done on the wavelet transformed data. In Chapter 4 we discuss some of them. One of the most important image processing operations is image compression. It is obvious that large images need to be stored and handled in a compressed form. Wavelets can provide for this since wavelet-based techniques offer better compression rates than other techniques.

Chapter 5 discusses the design and features of *WAILI*, a software library that implements the integer wavelet transforms described in Chapter 3. We developed WAILI as a basic software library to provide wavelet transforms and wavelet-based operations on two-dimensional images. The library is written in C++. In application areas like Geographical Information Systems (GIS), image sizes are measured in gigabytes and even terabytes. In Chapter 6 we describe the extension of WAILI to block-based processing, which allows for the handling of such very large images and for a parallel implementation.

In Chapter 7 we show how the lifting scheme can be used to create a new kind of second-generation wavelets on a rectangular grid, based on a red-black blocking scheme. These wavelets have some appealing properties compared to tensor products of traditional one-dimensional wavelets.

Besides applications in image and signal processing, wavelets are also an important tool for the creation of low-dimensional models for dynamical systems. Instead of analyzing the original partial differential equations (PDEs) that describe the dynamical system, one can analyze a low-dimensional model that approximates the original equations. The Proper Orthogonal Decomposition (POD) allows to create a "best" basis for such a model, derived from solutions of the PDEs, but it is an expensive operation. In Chapter 8 we propose a method using biorthogonal wavelet packet compression to reduce the complexity of the POD. The resulting Approximate POD is evaluated by constructing a low-dimensional model for a large-scale dynamical system.

1.3 Situation of the Research

Most of this work was carried out in the scope of the project "Wavelet-Based Interactive Video Communication and Image Database Consulting" (ITA/950244). This project involves three partners (Eurosense, Barco, and K.U.Leuven) and has the following goals:

- The development of a software library for wavelet transforms, including support for very large images and images composed of subimages.
- The implementation of image compression and processing operations.
- The integration in a Geographical Information System (GIS).
- A hardware assisted implementation of the wavelet transform for real-time full-motion video.

The research results presented in Chapter 8, about the Approximate Proper Orthogonal Decomposition, predate the above project. It is related to the ongoing research for the analysis of periodic solutions in large-scale dynamical systems described by partial differential equations by other members of the research group Scientific Computing of the Computer Science department.